UDC 538.4

PROBLEM OF AN ELECTROHYDRODYNAMIC PROBE WHICH DOES NOT DISTURB THE DISTRIBUTION OF CURRENT DENSITY AND VOLUME CHARGE

PMM Vol. 36, №3, 1972, pp. 555-559
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(Received January 31, 1972)

The problem of an electrohydrodynamic probe maintained at a floating potential is considered under the assumption that its characteristic dimension is small, compared with the characteristic dimension of the region of flow. The shape of the probe obtained in the course of solution is such, that its introduction into the flow leaves the current density and volume charge distributions unperturbed. Perturbations in the value of the electrical potential are computed and the value of the floating potential of the probe determined.

An elementary theory of an electrodynamic probe based on one-dimensional solutions is given in [1]. Below, unlike in [1], a three-dimensional problem of a flow around the probe is solved, and the solution obtained used to find a relation connecting the floating potential of the probe and the potential at the point under investigation.

Let us consider a steady-state potential flow of a homogeneous, incompressible, non-viscous fluid in an infinite space, at the velocity at infinity equal to $V^* = (u_\infty, 0, 0)$. A volume charge is introduced into the uncharged fluid at the emitter (plane $x = 0$) and the fluid is discharged at the collector (plane $x = L$). In the region $0 < x < L$ the flow of a unipolarly charged fluid obeys the electrohydrodynamics equations [2]. The problem is solved under the assumption that no interaction takes place between the emitter and the collector on one hand, and the hydrodynamic flow on the other hand. This assumption is based on the experimental data available, which indicates that the interaction is quite insignificant (e. g. in [1] the transmittance of the grid electrodes is equal to 0.95). We shall use the Ohm's law $\mathbf{j}^* = q^* (\mathbf{V}^* + b\mathbf{E}^*)$ (where $b =$ const and denotes mobility) and the fact that both, the electric field $\mathbf{E}^* = -\operatorname{grad}\varphi^*$, and the velocity field $\mathbf{V}^* = -\operatorname{grad}\Phi^*$, are potential. The dimensionless independent variables and unknown quantities are given by the formulas

$$x = L\xi, \quad y = L\eta, \quad z = L\zeta, \quad \varphi^* = u_\infty L\varphi / b,$$
$$\Phi^* = u_\infty L\Phi, \quad q^* = \varepsilon_0 u_\infty q / (4\pi bL)$$

We assume the electrohydrodynamic interaction parameter to be infinitely small. Then the system of electrohydrodynamics equations yields a system of equations for the volume charge, the electric and the hydrodynamic potentials

$$\Delta\Phi = 0, \quad \operatorname{div}[q \operatorname{grad}(\varphi + \Phi)] = 0, \quad \Delta\varphi = -q \tag{1}$$

Equation for the hydrodynamic potential can be solved independently from the other two equations, and the function Φ can be assumed known when the other two equations are being solved.

The boundary conditions for φ and q at the emitter and collector have the form

$$\varphi(0, \eta, \zeta) = 0, \quad \varphi(1, \eta, \zeta) = 0, \quad q(0, \eta, \zeta) = \infty \tag{2}$$

The last of these conditions corresponds [1] to a maximum current flowing from a unit area of the emitter (saturation current). As this quantity is finite, the condition $q(0, \eta, \zeta) = \infty$ becomes, in accordance with the Ohm's law, equivalent to the condition

$$\frac{\partial}{\partial\xi}\varphi(0, \eta, \zeta) + \frac{\partial}{\partial\xi}\Phi(0, \eta, \zeta) = 0 \tag{3}$$

Let us introduce the total potential $\chi = \varphi + \Phi$ and, having noted that $q = -\Delta\chi$, replace (1) by the following equivalent system of equations for Φ and χ

$$\Delta\Phi = 0, \quad \operatorname{div}[\Delta\chi \operatorname{grad}\chi] = 0 \tag{4}$$

The boundary conditions at the emitter and collector are obtained for χ from (2) and (3) and have the form

$$\chi(0, \eta, \zeta) = \Phi(0, \eta, \zeta), \quad \chi(1, \eta, \zeta) = \Phi(1, \eta, \zeta), \quad \frac{\partial}{\partial\xi}\chi(0, \eta, \zeta) = 0 \tag{5}$$

As we said before, when φ and therefore χ are determined, the function Φ can be assumed known.

We denote by G the region $0 < \xi < 1$, $-\infty < \eta < \infty$, $-\infty < \zeta < \infty$. A flow is called unperturbed, if all unknown quantities in G depend only on ξ, and we denote these quantities by a subscript o.

The dimensionless velocity at infinity of the hydrodynamic flow is (1, 0, 0), therefore the ξ-dependent solution of the first equation of (4) has the form

$$\Phi_0 = -\xi \tag{6}$$

The arbitrary additive constant is assumed to be zero. Solving the second equation of (4) with the boundary conditions (5) and taking (6) into account, we obtain $\chi_0 = -\xi^{3/2}$. Having obtained Φ_0 and χ_0, we can now find φ_0 and q_0

$$\varphi_0 = -\xi^{3/2} + \xi, \qquad q_0 = {}^3/_4\,\xi^{-1/2}$$

Let a convex conductor symmetrical with respect to the ξ-axis be placed between the emitter and the collector. We denote the region occupied by this body by Z, its surface by Σ and the region between $\xi = 0$ and $\xi = 1$ situated outside the body, by $G \setminus Z$. The region $G \setminus Z$ represents in this case a region of flow of a fluid containing a unipolar charge. In the dimensionless coordinates the distance between the left and the right end of the body is equal to ε, and the coordinates of the left end are $(\xi_0, 0, 0)$. The shape of the body, for the time being, remains unspecified.

Since the equation for Φ can be solved independently from the equations for χ and the hydrodynamic flow does not, by definition, interact with the electrodes, we can find Φ by solving the hydrodynamic flow around the body Z for the whole space and not only for the region $G \setminus Z$. In this case the boundary conditions for Φ become

$$\operatorname{grad}\Phi \to (1, 0, 0) \quad \text{as} \quad \xi^2 + \eta^2 + \zeta^2 \to \infty$$
$$\partial\Phi / \partial n\,|_\Sigma = 0$$

The problem can be solved for a body of an arbitrary surface shape Σ, therefore the following values can be found for Φ when $\xi = 0$, $\xi = 1$, and at the surface Σ

$$\Phi(0, \eta, \zeta) = \delta_1, \qquad \Phi(1, \eta, \zeta) = -1 + \delta_2, \qquad \Phi|_\Sigma = \Phi_\Sigma$$

where δ_1 and δ_2 denote the perturbations in the values of Φ_0 when $\xi = 0$ and $\xi = 1$, in the case of a flow around Z in the region G. The perturbations depend on both the linear size of the body and its shape.

The boundary conditions for χ have the form

$$\chi(0, \eta, \zeta) = \delta_1, \quad \chi(1, \eta, \zeta) = -1 + \delta_2, \quad \frac{\partial}{\partial \xi}\chi(0, \eta, \zeta) = 0$$
$$\chi|_\Sigma = \Phi_\Sigma + \varphi_\Sigma, \quad \varphi_\Sigma = \text{const}$$

with the value of φ_Σ not yet fixed. The function Φ_Σ is also dependent on the linear size of the body and on its shape. We require that the following three conditions hold:

$$\varepsilon \ll 1, \ \varepsilon / \xi_0 = O(\varepsilon), \ \varepsilon / (1 - \xi_0) = O(\varepsilon) \tag{7}$$

and seek a linear approximation in ε for the unknown quantities which we shall denote by the subscript 1.

Let us estimate δ_1 and δ_2 with respect to ε. By virtue of the conditions (7), in a three-dimensional space, δ_1 and δ_2 are of the order of ε^3. This follows from the fact that such estimates are valid for a sphere, while for a body of an arbitrary shape and a characteristic dimension $\sim\varepsilon$ the order of perturbations of the hydrodynamic potential does not depend on the shape of the body when the distances involved are large compared with ε [3]. Consequently $\delta_1 = 0$ and $\delta_2 = 0$ within the limits of the approximation considered. We can now write the boundary conditions for χ_1, and they have the following form:

$$\chi_1(0, \eta, \zeta) = 0, \ \chi_1(1, \eta, \zeta) = -1$$
$$\frac{\partial}{\partial \xi}\chi_1(0, \eta, \zeta) = 0, \quad \chi_1|_\Sigma = \Phi_{1\Sigma} + \varphi_{1\Sigma}$$

The second equation of (4) can be used to obtain χ_1, since it differs from the exact expression for χ_1 only in the terms of the order higher than ε and is therefore equivalent to it within the limits of the approximation used.

We now formulate the problem: to find a surface Σ of such shape that $\chi_{1\Sigma} = \Phi_{1\Sigma} + \varphi_{1\Sigma}$ would coincide with $\chi_0 = \Phi_0 + \varphi_0$ at some mentally chosen surface Σ in an unperturbed flow. If such a shape is found, then $\chi_1 \equiv \chi_0$ in the region $G \setminus Z$ since χ_1 and χ_0 satisfy the same equation and the same boundary conditions. If Φ_1 has been found, then the values of the perturbed electrical potential can be obtained from the formula

$$\varphi_1 = \chi_0 - \Phi_1 = -\xi^{3/2} - \Phi_1$$

The quantities q_1 and $\mathbf{j}_1$ remain unperturbed within the limits of the approximation considered

$$q_1 = -\Delta\chi_1 = -\Delta\chi_0 = q_0$$
$$\mathbf{j}_1 = q_1 \operatorname{grad}\chi_1 = q_0 \operatorname{grad}\chi_0 = \mathbf{j}_0$$

Let us now determine Φ_1, the shape of the surface Σ and the values of the constant $\varphi_{1\Sigma}$. We assume that such a surface has been found. In this case Φ has the following form at the surface of the streamlined body:

$$\Phi|_\Sigma = \Phi_\Sigma = \chi_0 - \varphi_\Sigma = -\xi^{3/2} - \varphi_\Sigma, \quad \varphi_\Sigma = \text{const} \tag{8}$$

Retaining in the expression for Φ_Σ only the terms linear in ε and expanding it in ε near the point $(\xi_0 + \varepsilon/2, 0, 0)$, we obtain

$$\Phi_1|_\Sigma = \Phi_{1\Sigma} = -\xi_0^{3/2} - {}^3/_4\xi_0^{1/2}\varepsilon - \varphi_{1\Sigma} - {}^3/_2\xi_0^{1/2}(\xi - \xi_0 - \varepsilon/2) \tag{9}$$
$$\xi - \xi_0 - \varepsilon/2 \sim \varepsilon$$

In this manner we arrive at the problem of determining the harmonic potential Φ_1 of a flow with specified velocity at infinity (1, 0, 0), which is a linear function of ξ at the streamlined surface Σ. As we know [4], in a flow around an ellipsoid the potential at the streamlined surface varies linearly. Let us therefore seek Σ in the form of an ellipsoid

$$\frac{(\xi-\xi_0-\varepsilon/2)^2}{(\varepsilon/2)^2}+\frac{\eta^2}{(\varepsilon\lambda/2)^2}+\frac{\zeta^2}{(\varepsilon\mu/2)^2}=1$$

Solving the problem of the flow around an ellipsoid [4], we obtain the following expression for Φ_1:

$$\Phi_1=-\left(\xi-\xi_0-\frac{\varepsilon}{2}\right)\left(1+\frac{a}{2-a_*}\right)-\xi_0-\frac{\varepsilon}{2} \tag{10}$$

$$a(\omega)=\lambda\mu\int_\omega^\infty[(1+u)(\lambda^2+u)(\mu^2+u)]^{-1/2}\frac{du}{1+u}, \quad a_*=a(0) \tag{11}$$

where a_* denotes the value of a at the ellipsoid surface and ω denotes the positive root of the following equation, for the points outside the ellipsoid:

$$\frac{(\xi-\xi_0-\varepsilon/2)^2}{1+\omega}+\frac{\eta^2}{\lambda^2+\omega}+\frac{\zeta^2}{\mu^2+\omega}=\left(\frac{\varepsilon}{2}\right)^2 \tag{12}$$

The additive constant is chosen so that $\Phi(0,\eta,\zeta)=0$ with the accuracy of up to the terms of the order higher than ε. From (10) we find that the streamlined surface Σ, $\Phi_{1\Sigma}$ has the form

$$\Phi_{1\Sigma}=-2(\xi-\xi_0-\varepsilon/2)/(2-a_*)-\xi_0-\varepsilon/2 \tag{13}$$

We thus have two expressions for the potential Φ_1 at the surface Σ, (9) and (13), for the same hydrodynamic flow, and these expressions must be equal to each other. Equating the coefficients of the zeroth and first power of $\xi-\xi_0-\varepsilon/2$ in the Taylor series of (9) and (13), we obtain $\varphi_{1\Sigma}$ and a_*

$$\varphi_{1\Sigma}=\xi_0-\xi_0^{3/2}+\varepsilon/2-{}^3/_4\xi_0^{1/2}\varepsilon, \qquad a_*=2-{}^4/_3\xi_0^{-1/2} \tag{14}$$

Comparing the expressions for a_* given in (11) and (14), we find the relation connecting λ, μ and ξ_0

$$a(0)=2-{}^4/_3\xi_0^{-1/2} \tag{15}$$

Thus the set of λ, μ pairs suitable in constructing a solution,forms a one-parameter family. The positivity of the left hand part of (15) implies that such solution can only be constructed in the region of G in which $4/9<\xi_0<1$. The electric potential decreases in this region together with the hydrodynamic potential, and this is the necessary condition without which such a solution could not be constructed.

To simplify the computations, we consider the case of an axially symmetric surface Σ, i.e. the case when $\lambda=\mu$. From (15) we obtain the following relation connecting λ and ξ_0

$$\frac{\lambda^2}{1-\lambda^2}\left[\frac{1}{\sqrt{1-\lambda^2}}\ln\frac{1+\sqrt{1-\lambda^2}}{1-\sqrt{1-\lambda^2}}-2\right]=2-\frac{4}{3}\xi_0^{-1/2}$$

It can be shown that λ increases monotonously from zero to unity when ξ_0 increases from $4/9$ to unity. We also obtain

$$a=\frac{\lambda^2}{1-\lambda^2}\left[\frac{1}{\sqrt{1-\lambda^2}}\ln\frac{\sqrt{1+\omega}+\sqrt{1-\lambda^2}}{\sqrt{1+\omega}-\sqrt{1-\lambda^2}}-\frac{2}{\sqrt{1+\omega}}\right]$$

where ω satisfies Eq. (12) in which $\lambda=\mu$.

Let us find the relative perturbation of the electric potential

$$\frac{\varphi - \varphi_0}{\varphi_0} = \frac{(\xi - \xi_0 - \varepsilon/2)\,a}{(\xi - \xi_0^{3/2})(2 - a_*)} \sim \frac{(\xi - \xi_0 - \varepsilon/2)\,\xi_0^{1/2}\lambda^2}{2(\xi - \xi_0^{3/2})(1 + \omega)^{3/2}}$$

The above estimate is valid for large ω, and this corresponds to large values of $[(\xi - \xi_0 - \varepsilon/2)^2 + \eta^2 + \zeta^2]^{1/2}$.

The corresponding formulas for the two-dimensional case can be obtained analogously; they are

$$a = \frac{2\lambda}{\sqrt{1-\lambda^2}}\left(\frac{\pi}{2} - \operatorname{arctg}\sqrt{\frac{1+\omega}{1-\lambda^2} - 1}\right), \qquad a_* = \frac{2\lambda}{\sqrt{1-\lambda^2}}\operatorname{arctg}\frac{\sqrt{1-\lambda^2}}{\lambda}$$

$$\frac{(\xi - \xi_0 - \varepsilon/2)^2}{1+\omega} + \frac{\eta^2}{\lambda^2 + \omega} = \left(\frac{\varepsilon}{2}\right)^2$$

$$\frac{\lambda}{\sqrt{1-\lambda^2}}\operatorname{arctg}\frac{\sqrt{1-\lambda^2}}{\lambda} = 1 - \frac{2}{3}\xi_0^{-1/2}, \qquad 0 < \lambda < 0.245$$

If a metallic probe,the surface of which coincides with Σ ,is inserted into an unperturbed flow at the point $(\xi_0 + \varepsilon/2,\ 0,\ 0)$ then a steady state will establish itself after a certain period of time (this was shown in [1]) and the probe will acquire a certain potential $\varphi_{\sim}^*$ (floating potential) which can be measured by means of a voltmeter. In this case the total current flowing onto the probe is zero, and the probe will work only if the last condition is fulfilled.

The solution constructed above represents the pattern of interaction between the probe and the flow, under the assumption that the solution of the electrohydrodynamic equations in unique. In particular, the solution φ_Σ^* obtained represents the value of the floating potential. Simple calculations show that in an unperturbed flow $\varphi_{\sim}^*$ coincides with φ_0^* at the point $(\xi_0 + \varepsilon/2,\ 0,\ 0)$. This means that, having the floating potential of such a probe, we can find the value of the potential in an unperturbed flow at a point which coincides with the center of the probe. In the solution obtained above, the requirement that the total current flowing into the probe is equal to zero is fulfilled automatically, since the current density and volume charge distributions both remain unperturbed within the limits of the accepted accuracy.

Analogous solutions can be obtained when the emitter and collector potentials are no longer equal to each other. In this case the region within which such a solution can be constructed is changed.

The problem of designing a probe which would not affect the current density and volume charge distributions similar to that considered above can be formulated, without the restricting conditions that ε is small and the region of flow specified. However, the problem considered above indicates that such a solution does not always exist.

The author thanks A. B. Vatazhin and G. A. Liubimov for their attention to this work.

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Translated by L. K.

UDC 538.4

EXACT SOLUTION OF THE PROBLEM OF AN INFINITELY CONDUCTING SPHERE WITH AN ARBITRARY VARYING RADIUS IN AN EXTERNAL MAGNETIC FIELD

PMM Vol.36, №3, 1972, pp. 560-561
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(Received July 17, 1969)

The variation in the elctric $E = \{0, 0, E_\varphi\}$ and magnetic $H = \{H_r, H_\vartheta, 0\}$ fields caused by a motion of an infinitely conductive sphere through a homogenous magnetic field $H_0 = \{H_0 \cos\vartheta, -H_0 \sin\vartheta, 0\}$, when the radius a of the sphere varies with time in a prescribed manner, was investigated in [1, 2]. Below we consider the same problem for the case when the dependence $a = a(t)$ is arbitrary.

The problem is reduced to solution of the Maxwell equations satisfying the following initial and boundary conditions

$$H_r(r, \vartheta, 0) = H_0 \cos\vartheta, \quad H_\vartheta(r, \vartheta, 0) = -H_0 \sin\vartheta, \quad E_\varphi(r, \vartheta, 0) = 0$$
$$H_r(a(t), \vartheta, t) = 0, \quad E_\varphi(a(t), \vartheta, t) - a^{\cdot} c^{-1} H_\vartheta(a(t), \vartheta, t) = 0 \qquad (1)$$

The last of these relations represents the usual electrodynamic condition [3] at the surface of an infinitely conducting sphere. For convenience, in the following, we shall replace the functions H_r, H_ϑ and E_φ by a single function $u(r, t)$ satisfying the wave equation and the conditions

$$u(r, 0) = {}^3/_2, \qquad (\partial u / \partial t)_{t=0} = 0$$
$$\left(\frac{\partial u}{\partial r}\right)_{r=a} + \frac{a^{\cdot}}{c^2}\left(\frac{\partial u}{\partial t}\right)_{r=a} + \frac{1}{c^2}\frac{d}{dt}[a^{\cdot} u(a, t)] + 2\frac{a^{\cdot 2}}{c^2}\frac{u(a, t)}{a} = 0$$

in the region $\{a(t) \leqslant r < \infty, t \geqslant 0\}$. This yields the following expressions for the unknown functions:

$$H_r(r, \vartheta, t) = \frac{2H_0 \cos\vartheta}{r^3}\int_{a(t)}^{r} u(x, t)\, x^2\, dx$$
$$H_\vartheta(r, \vartheta, t) = -H_0 u(r, t)\sin\vartheta + \frac{H_0 \sin\vartheta}{r^3}\int_{a(t)}^{r} u(x, t)\, x^2\, dx \qquad (2)$$
$$E_\varphi(r, \vartheta, t) = -\frac{H_0 \sin\vartheta}{r^2}\int_{a(t)}^{r}\frac{1}{c}\frac{\partial u}{\partial t} x^2 dx + \frac{a^{\cdot}}{c} H_0 \sin\vartheta\, \frac{a^2}{r^2} u(a, t)$$

The lower limit of integration in (2) is chosen with the boundary conditions (1) taken into account.